B. Sc. Part 2 (Hons.) 4th Paper, Diff. equations

Linear Diff. Equations with constant coefficients

1. $(D^2-4D+4)y = x^2$

Soln. For CF,

$D^2-4D+4=0 \Rightarrow (D-2)^2=0$

$\therefore D = 2, 2.$

$\Rightarrow CF = (c_1 + c_2 x)e^{2x}.$

For PI, $y = \dfrac{1}{D^2-4D+4}x^2$

$\Rightarrow y = \dfrac{1}{(D-2)^2}x^2 = \dfrac{1}{4\left(-\frac{D}{2}+1\right)^2}x^2$

$= \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2$

$= \frac{1}{4}\left(1+2\times\frac{D}{2}+\frac{3}{4}D^2+\cdots\right)x^2$

$\Rightarrow y = \frac{1}{4}\left(1+D+\frac{3}{4}D^2+\cdots\right)x^2$

$= \frac{1}{4}\left[x^2 + D(x^2) + \frac{3}{4}D^2(x^2)+\cdots\right]$

$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}\times 2 + 0\right]$

$\Rightarrow y = \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$

$\therefore$ complete soln is given by $y = CF + PI$

$\Rightarrow y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$ Ans

2. Solve $(D^2+D-6)y = x$.

Soln

For CF

$D^2+D-6=0 \Rightarrow (D+3)(D-2)=0$

$\therefore D = -3, 2$.

$\therefore CF = c_1 e^{-3x} + c_2 e^{2x}$

For PI

$$PI = \frac{1}{D^2+D-6} x = \frac{-1}{6\left(1-\frac{D}{6}-\frac{D^2}{6}\right)} x$$

$$= -\frac{1}{6}\left[1-\left(\frac{D}{6}+\frac{D^2}{6}\right)\right]^{-1} x$$

$$= -\frac{1}{6}\left[1+\left(\frac{D}{6}+\frac{D^2}{6}\right)+\left(\frac{D}{6}+\frac{D^2}{6}\right)^2+\cdots\right]x$$

$$= -\frac{1}{6}\left[x+\frac{D(x)}{6}+0\right]$$

$$\Rightarrow PI = -\frac{1}{6}\left[x+\frac{1}{6}\right] = -\frac{1}{36}(6x+1)$$

Hence, the complete solution is given by

$y = CF + PI$

$$\Rightarrow y = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{36}(6x+1)$$

Q. Solve $\left(\frac{d^2y}{dx^2} - 4y\right) = x^3$.

Soln The given equation is

$$\left(\frac{d^2}{dx^2} - 4\right) y = x^3$$

$$\Rightarrow (D^2 - 4) y = x^3.$$

For CF

$$D^2 - 4 = 0 \Rightarrow D^2 = 4$$

$$\therefore D = \pm 2$$

$\therefore CF = Ae^{2x} + Be^{-2x}$ where A and B are two arbitrary constants.

for PI

$$PI = \frac{1}{D^2 - 4} x^3 = -\frac{1}{4\left(1 - \frac{D^2}{4}\right)} x^3$$

$$\Rightarrow PI = -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} x^3$$

$$\Rightarrow PI = -\frac{1}{4}\left(1 + \frac{D^2}{4} + \frac{D^4}{16} + \cdots\right) x^3$$

$$\Rightarrow PI = -\frac{1}{4}\left[x^3 + \frac{D^2(x^3)}{4} + \frac{D^4(x^3)}{16} + \cdots\right]$$

$$\Rightarrow PI = -\frac{1}{4}\left[x^3 + \frac{6x}{4} + 0\right] = -\frac{1}{16}(4x^3 + 6x)$$

$\therefore$ Complete soln is $y = Ae^{2x} + Be^{-2x} - \frac{1}{16}(4x^3 + 6x)$

Q. Solve $(D^2 - 3D + 2)y = x^2 + x$.

Soln For CF, $D^2 - 3D + 2 = 0 \Rightarrow D = 2, 1$.

$\therefore CF = Ae^{x} + Be^{2x}$.

Now, $PI = \dfrac{1}{D^2 - 3D + 2}(x^2 + x)$

$$\Rightarrow PI = \frac{1}{2\left(1 - \frac{3}{2}D + \frac{1}{2}D^2\right)}(x^2 + x)$$

$$= \frac{1}{2}\left[1 - \left(\frac{3}{2}D - \frac{1}{2}D^2\right)\right]^{-1}(x^2 + x)$$

$$= \frac{1}{2}\left[1 + \left(\frac{3}{2}D - \frac{1}{2}D^2\right) + \left(\frac{3}{2}D - \frac{1}{2}D^2\right)^2 + \text{higher powers of } D\right](x^2 + x)$$

$$\Rightarrow PI = \frac{1}{2}\left[(x^2 + x) + \frac{3}{2}D(x^2 + x) - \frac{1}{2}D^2(x^2 + x) + \frac{9}{4}D^2(x^2 + x) + \text{higher powers of } D\right]$$

$$= \frac{1}{2}\left[x^2 + x + \frac{3}{2}(2x + 1) - \frac{1}{2}(2) + \frac{9}{4} \times 2 + 0\right]$$

$$\Rightarrow PI = \frac{1}{2}\left[x^2 + 4x + 5\right]$$

Hence, the complete solution is given by

$$y = CF + PI$$

$$\Rightarrow y = Ae^{x} + Be^{2x} + \frac{1}{2}(x^2 + 4x + 5).$$